Gaussian fluctuations in the two-dimensional BCS-BEC crossover

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Outline

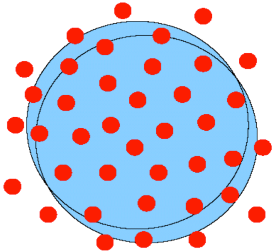
- Introduction and motivation: BCS-BEC crossover in 2D.
- Theoretical description of a 2D Fermi gas using the path integral formalism: mean-field and Gaussian fluctuations.
- The role of fluctuations: the composite boson limit.
- Results and comparison with experimental data:
    - Equation of state
    - First sound
    - Second sound
    - Berezinskii-Kosterlitz-Thouless critical temperature.

Main reference: GB and L. Salasnich, Phys. Rev. B **93**, 014519 (2016).

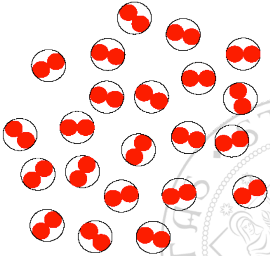
The BCS-BEC crossover (1/2)

In 2004 the **BCS-BEC crossover** has been observed with ultracold gases made of fermionic ^{40}K and ^{6}Li alkali-metal atoms. The fermion-fermion attractive interaction can be tuned (using a Feshbach resonance), from weakly to strongly interacting.

BCS regime: weakly interacting Cooper pairs.

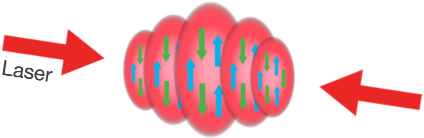


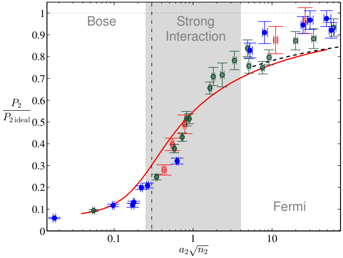
BEC regime: tightly bound bosonic molecules.



The BCS-BEC crossover (2/2)

An additional laser confinement can be used to create a quasi-2D pancake geometry, trapping the fermions in the antinodes of a standing optical wave.





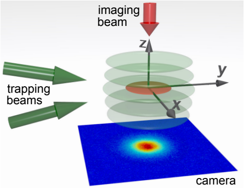
In 2014 the 2D BCS-BEC crossover has been observed[1] with a **quasi-2D Fermi gas of ^{6}Li atoms** with widely tunable s-wave interaction. The pressure P versus the gas parameter $a_B\sqrt{n}$ has been measured.

[1] V. Makhalov, K. Martiyanov, and A. Turlapov, PRL **112**, 045301 (2014).

The BCS-BEC crossover in 2D (1/2)

Many properties of 2D Fermi condensates are currently being studied:

- **Imaging of the atomic cloud**[1]**.**
- Equation of state.
- Recently (June 2015) the direct observation of the BKT transition has been reported[2].
- Dynamic properties: sound velocity.



[1] M. G. Ries et al., Phys. Rev. Lett. **114**, 230401 (2015)

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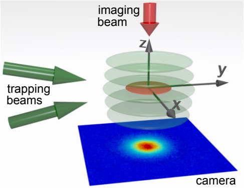

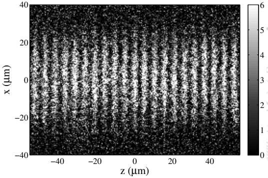

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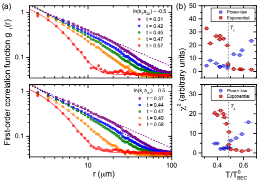

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- Qualitatively new physics: a bound state is always present.
- Berezinskii-Kosterlitz-Thouless mechanism:
  - Mermin-Wagner-Hohenberg theorem: no condensation at finite temperature, no off-diagonal long-range order.
  - Algebraic decay of correlation functions $\langle \exp(\mathrm{i}\theta(\mathbf{r})) \exp(\mathrm{i}\theta(0)) \rangle \sim |\mathbf{r}|^{-\eta}$
  - Transition to the normal state at a finite temperature T_{BKT}.

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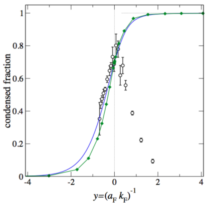

Example: theoretically-derived 3D condensate fraction as a function of $y = (k_F a_s)^{-1}$ compared to experimental data (MIT, Zwierlein group). From L. Salasnich et al., PRA **72**, 023621 (2005).

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- The fluctuations are more relevant for lower dimensionalities. The mean field theory can correctly describe (to some extent) the crossover in 3D, we expect it not to work at all in 2D.
- The physics of the BCS-BEC crossover is also relevant in the description of many different systems (bilayers of dipolar atoms, exciton-polariton condensates, thin ^{3}He films). It may also be relevant for the description of high-T_c cuprates as the scaled correlation length ($k_F\xi_0 \sim 5$ for YBCO and $k_F\xi_0 \sim 10$ for LSCO) lies between the BCS ($k_F\xi_0 \sim 10^3$) and BEC ($k_F\xi_0 \ll 1$) regimes.

Path integral description of a Fermi gas (1/4)

The partition function $\mathcal{Z}$ of a uniform system at temperature T, in a d-dimensional volume L^d, and with chemical potential μ reads

$$\mathcal{Z} = \int \mathcal{D}\psi_\sigma \mathcal{D}\bar{\psi}_\sigma e^{-S[\psi_\sigma, \bar{\psi}_\sigma]}$$

Fermions are described by anticommuting Grassmann fields, $\psi_\sigma(\mathbf{x}, \tau)$ and the imaginary time goes from 0 to $\hbar\beta$, where $\beta = \frac{1}{k_B T}$. Action:

$$S = S_{\text{free}} + S_{\text{int}}$$

- Action for a free particle:

$$S_{\text{free}}[\psi_\sigma, \bar{\psi}_\sigma] = \int_0^{\hbar\beta} \mathrm{d}\tau \int \mathrm{d}^d x \sum_\sigma \bar{\psi}(\mathbf{x}, \tau) \left[\hbar \frac{\partial}{\partial \tau} - \frac{\hbar^2}{2m}\nabla^2 - \mu\right] \psi(\mathbf{x}, \tau)$$

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- Interaction term:

$$S_{\text{int}}[\psi_\sigma,\bar{\psi}_\sigma] = \int_0^{\hbar\beta} \mathrm{d}\tau \int \mathrm{d}^d x \mathrm{d}^d y\, \bar{\psi}_\uparrow(\mathbf{x},\tau)\bar{\psi}_\downarrow(\mathbf{y},\tau) V(\mathbf{x}-\mathbf{y}) \psi_\downarrow(\mathbf{y},\tau)\psi_\uparrow(\mathbf{x},\tau)$$

For a dilute gas one can use $V(\mathbf{x}-\mathbf{y}) = g_0\delta(\mathbf{x}-\mathbf{y})$, where $g_0 < 0$ is the attractive strength of the s-wave coupling.

Path integral description of a Fermi gas (2/4)

How to treat the quartic interaction term $\sim \psi^4$?

- We use a Hubbard-Stratonovich transformation, introducing the auxiliary field $\Delta(x)$ and the shorthand $x = (\mathbf{x}, \tau)$.
- The interaction between fermions is described in terms of an exchange boson.

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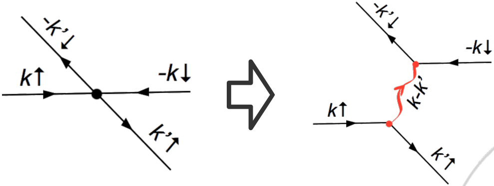

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$$e^{-g_0 \int \mathrm{d}x \; \bar{\psi}_\uparrow(x)\bar{\psi}_\downarrow(x)\psi_\downarrow(x)\psi_\uparrow(x)} \propto \int \mathcal{D}\Delta\mathcal{D}\bar{\Delta} \; e^{\int \mathrm{d}x \; \left(\frac{|\Delta|^2}{g_0} + \bar{\Delta}\psi_\downarrow\psi_\uparrow + \Delta\bar{\psi}_\uparrow\bar{\psi}_\downarrow\right)}$$

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Remarks:

- Essentially a Gaussian integral.
- Physical meaning of the transformation: $\Delta \sim \psi\psi$, as in the BCS theory a finite expectation value signals the onset of pairing.
- Result: the quartic interaction is decoupled, but we have introduced a new field, hopefully we can treat in perturbatively.

Path integral description of a Fermi gas (3/4)

After the H/S transformation the partition function can be recast in an elegant way using the Nambu-Gor'kov spinors ($\Psi(x) = \begin{pmatrix}\psi_\uparrow(x) & \bar{\psi}_\downarrow(x)\end{pmatrix}^T$)

$$\mathcal{Z} = \int \mathcal{D}\Delta\mathcal{D}\bar{\Delta}\mathcal{D}\psi_\sigma\mathcal{D}\bar{\psi}_\sigma \exp\left[\int \mathrm{d}x \, (\bar{\Psi}(x)\left[-\mathbb{G}^{-1}\right]_x \Psi(x) - \frac{|\Delta(x)|^2}{g_0})\right]$$

The integration over the fermionic fields $\psi_\sigma, \bar{\psi}_\sigma$ can now be carried out exactly, being the action quadratic form in the fermionic fields, yielding:

$$\mathcal{Z} = \int \mathcal{D}\Delta\mathcal{D}\bar{\Delta} \exp\left[\mathrm{Tr}\ln\left(-\mathbb{G}^{-1}\right) + \int \mathrm{d}x \, \frac{|\Delta|^2}{g_0}\right]$$

The complete physics of the system is encoded in the Green's function $\mathbb{G}$.

$$\left[-\mathbb{G}^{-1}\right]_x = \begin{pmatrix}\hbar\partial_\tau + \xi & -\Delta(x) \\ -\bar{\Delta}(x) & \hbar\partial_\tau - \xi\end{pmatrix}$$

with $\xi = -\frac{\hbar^2\nabla^2}{2m} - \mu$.

The correlation functions generating functional

Let us consider for simplicity the case with a constant, uniform Δ. Let us construct a generating functional by adding source terms to the partition function $\mathcal{Z} = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\exp\left(-\bar{\psi}_i\mathbb{G}^{-1}_{ij}\psi_j\right)$ and integrating out the fermionic fields (i, j can be any index: position, spin, Nambu space)

$$\mathcal{Z}[J,\bar{J}] = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\, e^{-\bar{\psi}_i\mathbb{G}^{-1}_{ij}\psi_j + \bar{J}_i\psi_i + J\bar{\psi}_i} = \det\left(\mathbb{G}^{-1}_{ij}\right)e^{\bar{J}_i\mathbb{G}_{ij}J_j}$$

Differentiating twice w.r.t. the sources ones sees that the Green function $\mathbb{G}$ is the 2-point function

$$\frac{1}{\mathcal{Z}}\,\frac{\delta^2}{\delta J_a \delta \bar{J}_b}\mathcal{Z}\bigg|_{J=\bar{J}=0} = \langle \psi_a \bar{\psi}_b \rangle = \mathbb{G}_{ab}$$

and by applying the same strategy multiple times and subsequently using Wick's theorem one can derive n-point functions in terms of $\mathbb{G}$.

In the present case the Green's function $\mathbb{G}$ is a 2×2 matrix in Nambu space, and corresponds to the (imaginary-) time-ordered expectation values for fermionic fields, here G

$$\mathbb{G}(x) = \begin{pmatrix} \langle T_\tau \psi(x) \psi^\dagger(0) \rangle & \langle T_\tau \psi(x) \psi(0) \rangle \\ \langle T_\tau \psi^\dagger(x) \psi^\dagger(0) \rangle & \langle T_\tau \psi^\dagger(x) \psi(0) \rangle \end{pmatrix} = \begin{pmatrix} G & F \\ \bar{F} & \bar{G} \end{pmatrix}$$

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$$\mathcal{Z} = \int \mathcal{D}\Delta \mathcal{D}\bar{\Delta} \exp\left[\mathrm{Tr}\ln\left(-\mathbb{G}^{-1}\right) + \int \mathrm{d}x \, \frac{|\Delta|^2}{g_0}\right]$$

Path integral description of a Fermi gas (4/4)

How to tackle the problem? Idea: separate a leading (and integrable) contribution from a small contribution (to be treated perturbatively).
We expand the pairing field Δ around a constant and uniform saddle-point configuration Δ_0, as

$$\Delta(x) = \Delta_0 + \eta(x)$$

Mean field
Fluctuations

it follows that

$$\underbrace{\begin{pmatrix} \hbar\partial_\tau + \xi & -\Delta(x) \\ -\bar{\Delta}(x) & \hbar\partial_\tau - \xi \end{pmatrix}}_{[-\mathbb{G}^{-1}]_x} = \underbrace{\begin{pmatrix} \hbar\partial_\tau + \xi & -\Delta_0 \\ -\bar{\Delta}_0 & \hbar\partial_\tau - \xi \end{pmatrix}}_{\left[-\mathbb{G}_{\mathrm{sp}}^{-1}\right]_x} + \underbrace{\begin{pmatrix} 0 & -\eta(x) \\ -\bar{\eta}(x) & 0 \end{pmatrix}}_{[\mathbb{F}]_x}$$

Mean field and fluctuations (1/2)

We separate the mean-field and fluctuations components in the partition function, too! The $\ln(-\mathbb{G}^{-1})$ becomes

$$\ln\left(-\mathbb{G}^{-1}\right) = \ln\left(-\mathbb{G}_{\mathrm{sp}}^{-1}\right) + \ln\left(\mathbb{1} - \mathbb{G}_{\mathrm{sp}}\mathbb{F}\right)$$

expanding the logarithm up to the second order in the fluctuation fields η we obtain

$$\ln\left(\mathbb{1} - \mathbb{G}_{\mathrm{sp}}\mathbb{F}\right) = -\sum_{n=1}^{\infty} \frac{\left(\mathbb{G}_{\mathrm{sp}}\mathbb{F}\right)^n}{n} \approx \cancel{-\mathbb{G}_{\mathrm{sp}}\mathbb{F}} - \frac{1}{2}\mathbb{G}_{\mathrm{sp}}\mathbb{F}\mathbb{G}_{\mathrm{sp}}\mathbb{F} + \mathcal{O}\left(\eta^3\right)$$

Final result: mean-field and Gaussian-level partition functions

$$\mathcal{Z} \approx \int \mathcal{D}\Delta\mathcal{D}\bar{\Delta}\, e^{\mathrm{Tr}\ln\left(-\mathbb{G}^{-1}\right)} = \int \mathcal{D}\Delta\mathcal{D}\bar{\Delta}\, e^{\mathrm{Tr}\ln\left(-\mathbb{G}_{\mathrm{sp}}^{-1}\right)} e^{\mathrm{Tr}\ln\left(\mathbb{1}-\mathbb{G}_{\mathrm{sp}}\mathbb{F}\right)} = \mathcal{Z}_{\mathrm{mf}}\,\mathcal{Z}_{\mathrm{fl}}$$

with:

$$\mathcal{Z}_{\mathrm{mf}} = \det\left(-\mathbb{G}_{\mathrm{sp}}^{-1}\right) \qquad \mathcal{Z}_{\mathrm{fl}} = \int \mathcal{D}\eta\mathcal{D}\bar{\eta}\; e^{-\frac{1}{2}\mathrm{Tr}\left(\mathbb{G}_{\mathrm{sp}}\mathbb{F}\mathbb{G}_{\mathrm{sp}}\mathbb{F}\right) + \int \mathrm{d}x \frac{|\eta|^2}{g_0}}$$

Mean field and fluctuations (2/2)

Using $\mathcal{Z} = e^{-\beta\Omega}$, where Ω is the thermodynamic grand potential, one gets the mean-field equation of state:

$$\Omega_{\mathrm{mf}}(\mu) = -\frac{mL^2}{2\pi\hbar^2}(\mu + \frac{1}{2}\epsilon_B)^2$$

where ϵ_B is the binding energy of a pair, and the Gaussian-level contribution to the grand potential ($Q = (\mathbf{q}, \mathrm{i}\Omega_n)$ and Ω_n are Bose Matsubara frequencies.):

$$\Omega_{\mathrm{fl}}(\mu, \Delta_0) = \frac{1}{2\beta}\sum_Q \ln\det(\mathbb{M}(Q))$$

- At mean-field the algebra is very simple, $\mathbf{k}$-integrations are analytical.
- The fluctuations algebra on the other hand is quite involved.

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$$M_{12}(\mathbf{q}, \mathrm{i}\Omega_m) = -\Delta_0^2 \sum_{\mathbf{k}} \frac{\tanh(\beta E_{\mathbf{k}}/2)}{2E_{\mathbf{k}}} \left[\frac{1}{(\mathrm{i}\Omega_m - E_{\mathbf{k}} + E_{\mathbf{k}+\mathbf{q}})(\mathrm{i}\Omega_m - E_{\mathbf{k}} - E_{\mathbf{k}+\mathbf{q}})} + \right.$$
$$\left. + \frac{1}{(\mathrm{i}\Omega_m + E_{\mathbf{k}} - E_{\mathbf{k}+\mathbf{q}})(\mathrm{i}\Omega_m + E_{\mathbf{k}} + E_{\mathbf{k}+\mathbf{q}})} \right]$$

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$$\xi_{\mathbf{k}} = \frac{\hbar^2 k^2}{2m} - \mu \qquad E_{\mathbf{k}} = \sqrt{\xi_{\mathbf{k}}^2 + \Delta_0^2}$$

Single particle and collective excitations

One finds that in the gas of paired fermions there are two kinds of elementary excitations: fermionic single-particle excitations with energy

$$E_{\mathrm{sp}}(k) = \sqrt{\left(\frac{\hbar^2 k^2}{2m} - \mu\right)^2 + \Delta_0^2} \ ,$$

where Δ_0 is the pairing gap, and bosonic collective excitations with energy

$$E_{\mathrm{col}}(q) = \sqrt{\frac{\hbar^2 q^2}{2m}\left(\lambda \frac{\hbar^2 q^2}{2m} + 2mc_s^2\right)} \ ,$$

where λ is the first correction to the familiar low-momentum phonon dispersion $E_{\mathrm{col}}(q) \simeq c_s \hbar q$ and c_s is the sound velocity.

The extended BCS equations

Gap equation

The saddle point condition determines Δ_0.

$$\frac{\partial \Omega}{\partial \Delta_0} = 0$$

Number equation

The number equation is used to implicitly determine μ as a function of the number of particles N.

$$N = -\frac{\partial \Omega}{\partial \mu} = -\frac{\partial \Omega_{\text{mf}}}{\partial \mu} - \frac{\partial \Omega_{\text{fl}}}{\partial \mu} - \frac{\partial \Omega_{\text{fl}}}{\partial \Delta_0}\frac{\partial \Delta_0}{\partial \mu}$$

The gap and number equation are jointly solved and determine the chemical potential μ and the pairing gap Δ_0 as a function of the crossover.

Bound state equation

In 2D the strength of the attractive s-wave potential is $g_0 < 0$, which can be implicitely related to the bound state energy:

$$-\frac{1}{g_0} = \frac{1}{2L^2} \sum_{\mathbf{k}} \frac{1}{\epsilon_k + \frac{1}{2}\epsilon_B} \; .$$

with $\epsilon_k = \hbar^2 k^2/(2m)$. In 2D, as opposed to the 3D case, a bound state exists even for arbitrarily weak interactions, making ϵ_B a good variable to describe the whole BCS-BEC crossover.

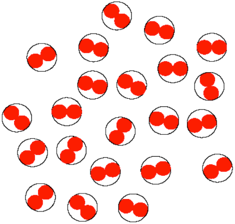
The binding energy ϵ_B and the fermionic (2D) scattering length a_{2D} are related by the equation[1]:

$$\epsilon_B = \frac{4\hbar^2}{e^{2\gamma} m a_{2D}^2}$$

[1] C. Mora and Y. Castin, Phys. Rev. A **67**, 053615 (2003).

The role of Gaussian fluctuations and collective excitations: composite bosons (1/3)

In the strongly interacting limit an attractive Fermi gas maps to a gas of composite bosons with chemical potential $\mu_B = 2(\mu + \epsilon_B/2)$ and mass $m_B = 2m$. Residual interaction between bosons.

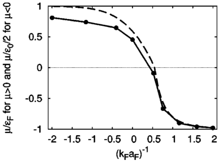


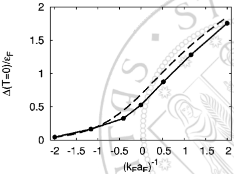
The present theory extends the BCS theory to the strong-coupling regime, following Leggett's original intuition. **One may want to check**: is the strong coupling limit correctly recovered at mean-field? And at a Gaussian level?

The role of Gaussian fluctuations and collective excitations: composite bosons (1/2)

In three dimensions:

- Even a mean-field theory may provide good precision in the intermediate and BEC regime at $T = 0$, e.g. in the case of the condensate fraction.
- Residual interaction between bosons: $a_b = 2a_s$ (mean-field), $a_b = {}^2\!/_3 a_s$ (Monte Carlo, fluctuations[1], experiments).
- Finite temperature properties: only in the strictly BCS limit.
- Equation of state (from P. Pieri et al., Phys. Rev. B **70**, 094508 (2004)); chemical potential μ and pairing gap Δ_0 across the crossover, MF vs FL.



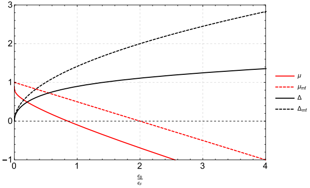


[1] L. Salasnich and GB, Phys. Rev. A **91**, 033610 (2015).

The role of Gaussian fluctuations and collective excitations: composite bosons (2/2)

In two dimensions:

- The role of the fluctations is, as expected, more relevant. The mean-field theory is expected to work just in strictly BCS limit.
- It has been shown that the inclusion of Gaussian fluctuations at $T = 0$ reproduces Popov's equation of state of 2D bosons, with $a_B = a_F/(2^{\frac{1}{2}} e^{\frac{1}{4}}) \simeq 0.551 a_F$, in agreement with Monte Carlo calculations.
- The equation of state of composite 2D bosons is radically different.



[1] L. Salasnich and F. Toigo, Phys. Rev. A **91**, 011604(R) (2015)

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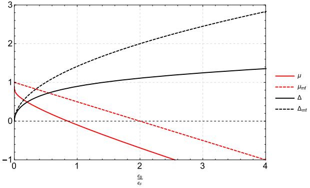


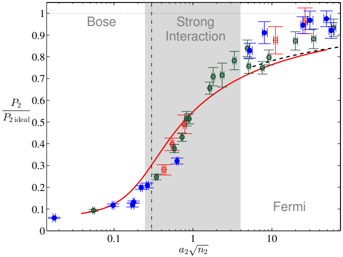
Gaussian fluctuations are crucial in correctly describing the properties of a 2D Fermi gas in the intermediate and strong-coupling regimes!

[1] L. Salasnich and F. Toigo, Phys. Rev. A **91**, 011604(R) (2015)

Theory vs. experiments

Equation of state

The pressure measured as a function of the adimensional gas parameter $a_B\sqrt{n_B}$. Experimental data, as shown in the introduction (red curve: smooth approximation of pure 2D Monte Carlo simulation) vs. the present model (gray dashed curve: mean-field, black curve: with fluctuations)



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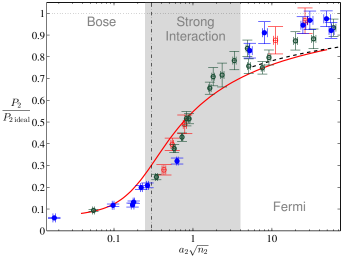

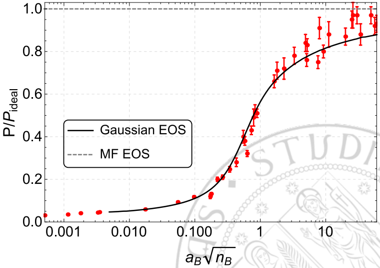

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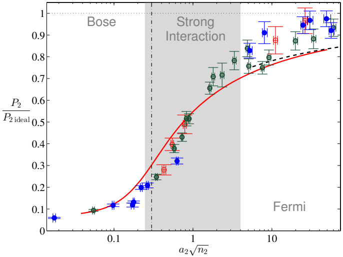
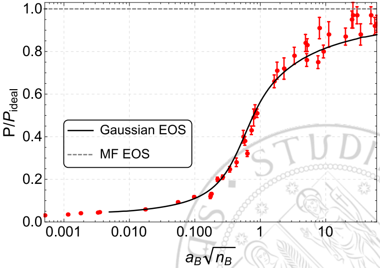




See also: L. He et al., Phys. Rev. A **92**, 023620 (2015).

First sound velocity (1/2)

The first sound velocity c_s can be read from the collective excitations spectrum:

$$E_{col}(q) = \sqrt{\frac{\hbar^2 q^2}{2m}\left(\lambda \frac{\hbar^2 q^2}{2m} + 2m{c_s}^2\right)}$$

Alternatively the $T = 0$ sound velocity is calculated through the thermodynamics formula:

$$c_s = \sqrt{\frac{n}{m}\frac{\partial \mu}{\partial n}}$$

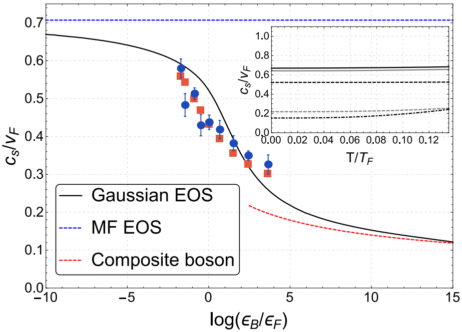
We compare our result with:

- The mean-field result, neglecting Gaussian fluctuations.
- The composite boson limit, obtained through Popov's equation of state for 2D interacting bosons

$$c_s^2 = \frac{4\pi\hbar^2}{m_B^2}\frac{n_B}{\ln\left(\frac{1}{n_B a_B^2}\right)}$$

- Preliminary experimental data (University of Hamburg).

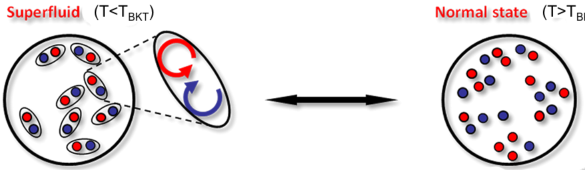
First sound velocity (2/2)



- Away from the weak-coupling limit, in the intermediate region and in the BEC limit the sound velocity c_s is strongly affected by the Gaussian contribution to the equation of state.
- Strong coupling: composite boson limit.
- Quite good agreement with (preliminary) experimental data.
- The temperature dependence (inset) is very weak.

BKT critical temperature (1/4)

The Berezinskii-Kosterlitz-Thouless (BKT) transition separated the low-temperature phase characterized by bound vortex-antivortex pairs from the high-temperature phase characterized by a proliferation of free vortices.



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$$k_B T_{BKT} = \frac{\hbar^2 \pi}{8m} n_s(T_{BKT})$$

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The superfluid density is obtained using Landau's quasiparticle excitations formula for fermionic and bosonic excitations:

$$n_{n,f} = \beta \int \frac{\mathrm{d}^2 k}{(2\pi)^2} k^2 \frac{e^{\beta E_k}}{(e^{\beta E_k} + 1)^2} \quad \text{and} \quad n_{n,b} = \frac{\beta}{2} \int \frac{\mathrm{d}^2 q}{(2\pi)^2} q^2 \frac{e^{\beta E_{col}}}{(e^{\beta E_{col}} - 1)^2},$$

then $n_s = n - n_{n,f} - n_{n,b}$.

BKT critical temperature (2/4)

Main approximation

The single-particle and collective contributions are not independent, as there is hybridization due to Landau damping at finite temperature.

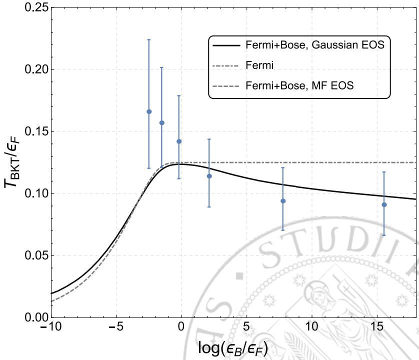
The effect of hybridization is most prominent at $T \sim \epsilon_F$, here in the superfluid phase, below T_{BKT}, one has $k_B T \lesssim 0.125\epsilon_F$ and the hybridization can be safely ignored.

Previous results *a posteriori* confirm that hybridization should be neglectable.

BKT critical temperature (3/4)

We can compare the theory with recently obtained experimental data[2]:

- The agreement with experimental data is very good in the intermediate and strongly coupled regimes.
- The agreement for two points in the weakly-coupled regime is not as good, but still within 1.2σ.
- However, under very general assumptions, $T_{BKT} \lesssim 0.125\epsilon_F$ if the Kosterlitz-Nelson condition holds.



[1]P. A. Murthy et al., Phys. Rev. Lett. **115**, 010401 (2015).

BKT critical temperature (4/4)

Composite boson limit: combining $a_B = \frac{1}{2^{1/2} e^{1/4}} a_F$, $\epsilon_B = \frac{4}{e^{2\gamma}} \frac{\hbar^2}{m a_F{}^2}$ we get:

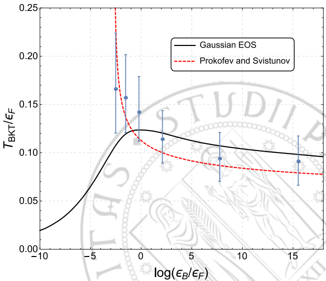
$$\frac{\epsilon_B}{\epsilon_F} = \frac{\kappa}{n_B a_B^2} \qquad \kappa \simeq 0.061$$

The strongly bound regime maps to the low density limit of a Bose gas.

Prokofev and Svistunov, using a mixed analytical and Monte Carlo approach, have found for 2D bosons:

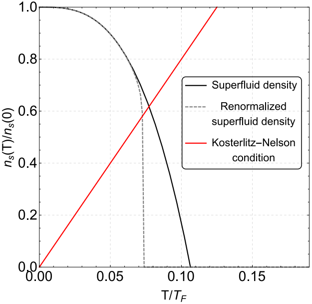
$$T_{BKT} = \frac{2\pi n_B}{m_B \log\left(\frac{\xi}{m_B U_{\text{eff}}}\right)} \qquad U_{\text{eff}} = \frac{4\pi}{m_B \log(1/n_B a_B^2)}$$

with $\xi \sim 380$. Putting everything all together one obtains an estimate for T_{BKT} valid in the strongly-coupled regime. How does it compares to the present theory and to experimental data?



Vortices

Preliminary study of the superfluid density renormalization due to the contribution of vortices, currently in progress.



Kosterlitz renormalization group equations

$$\begin{cases} \frac{dK^{-1}(l)}{dl} = 4\pi^3 y^2(l) + O(y^3) \\ \frac{dy(l)}{dl} = \left[2 - \pi K(l)\right] y(l) + O(y^2) \end{cases}$$

With initial conditions $K(l=0) = n_s(l)/T$ and $y(l=0) = \exp(-\pi n_s/2T)$, we calculate $K(\infty)$.

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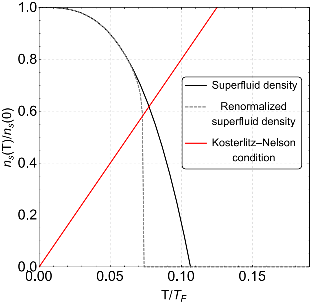
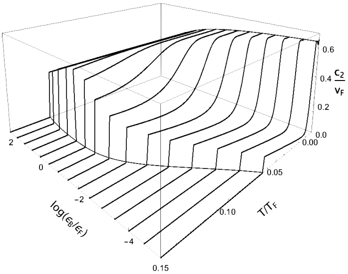


The BKT critical temperature is slightly lower, especially in the Bose regime, as an effect of the renormalized superfluid density.

Is it possible to calculate $\xi \sim 380$?

Second sound velocity



A superfluid can also sustain the second sound (entropy wave as opposed to density wave). Using the same approximation as before, we model the free energy as:

$$F_{sp} = -\frac{2}{\beta}\sum_{\mathbf{k}} \ln\left[1 + e^{-\beta E_{sp}(k)}\right]$$

$$F_{col} = \frac{1}{\beta}\sum_{\mathbf{q}} \ln\left[1 - e^{-\beta E_{col}(q)}\right]$$

The second sound velocity is readily calculated from the entropy as:

$$S = -(\partial F/\partial T)_{N,L^2} \qquad c_2 = \sqrt{\frac{1}{m}\frac{\bar{S}^2}{\left(\frac{\partial \bar{S}}{\partial T}\right)_{N,L^2}}\frac{n_s}{n_n}}$$

Conclusions

- A theoretical description of an interacting Fermi gas has been developed using a path integral formulation, consisting of a mean-field theory and of Gaussian fluctuations for the order parameter.
- It has been shown that the theoretical treatment of a 2D Fermi gas requires the inclusion of Gaussian fluctuations.
- This approach shows good agreement with experimental data (equation of state, BKT critical temperature, first sound), other predictions are open to verification (second sound).
- This treatment can be extended to 2D systems with BCS-like pairing (bilayers of polar molecules, exciton-polariton condensates, etc.)

Thanks for your attention.

(These slides are available at http://bighin.com)