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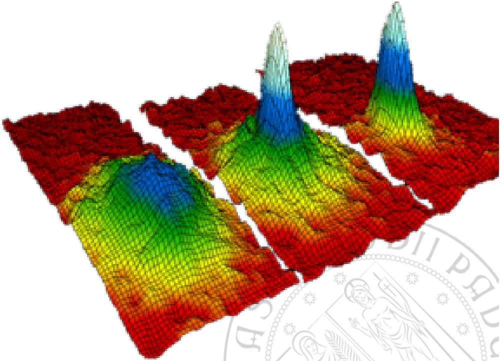
April 4th, 2016

Summary

- The BCS-BEC crossover in ultracold Fermi gases
  - The theoretical treatment: mean-field and fluctuations
  - The two-dimensional Fermi gas
    - Equation of state
    - First sound
    - BKT critical temperature
  - Beliaev decay of collective excitations
- Superconductivity in high-T_c cuprates: a gauge approach
  - Superfluid density

Ultracold Fermi gases

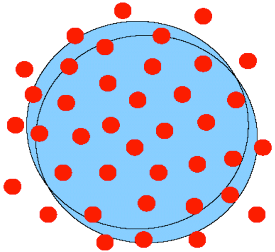
- Ultracold gases: experimental observation of quantum properties of matter. Vortices in a superfluid, BEC.
- Bose-Einstein condensation (1995), degenerate Fermi gas and fermionic condensate (2003).
- Very clean experimental environment: control over the temperature, the number of particles, the interaction.



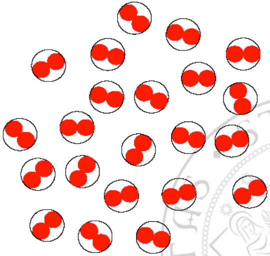
The BCS-BEC crossover

In 2004 the **BCS-BEC crossover** has been observed with ultracold gases made of fermionic ^{40}K and ^{6}Li alkali-metal atoms. The fermion-fermion attractive interaction can be tuned (using a Feshbach resonance), from weakly to strongly interacting.

BCS regime: weakly interacting Cooper pairs.



BEC regime: tightly bound bosonic molecules.



Path integral description of a Fermi gas (1/4)

The partition function $\mathcal{Z}$ of a uniform system at temperature T, in a d-dimensional volume L^d, and with chemical potential μ reads

$$\mathcal{Z} = \int \mathcal{D}\psi_\sigma \mathcal{D}\bar{\psi}_\sigma e^{-S[\psi_\sigma,\bar{\psi}_\sigma]}$$

Fermions are described by anticommuting Grassmann fields, $\psi_\sigma(\mathbf{x},\tau)$ and the imaginary time goes from 0 to $\hbar\beta$, where $\beta = \frac{1}{k_B T}$. Action:

$$S = S_{\text{free}} + S_{\text{int}}$$

- Action for a free particle:

$$S_{\text{free}}[\psi_\sigma,\bar{\psi}_\sigma] = \int_0^{\hbar\beta} \mathrm{d}\tau \int \mathrm{d}^d x \sum_\sigma \bar{\psi}(\mathbf{x},\tau) \left[\hbar\frac{\partial}{\partial\tau} - \frac{\hbar^2}{2m}\nabla^2 - \mu\right] \psi(\mathbf{x},\tau)$$

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- Interaction term:

$$S_{\text{int}}[\psi_\sigma, \bar{\psi}_\sigma] = \int_0^{\hbar\beta} \mathrm{d}\tau \int \mathrm{d}^d x \mathrm{d}^d y \, \bar{\psi}_\uparrow(\mathbf{x}, \tau)\bar{\psi}_\downarrow(\mathbf{y}, \tau) V(\mathbf{x}-\mathbf{y})\psi_\downarrow(\mathbf{y}, \tau)\psi_\uparrow(\mathbf{x}, \tau)$$

For a dilute gas one can use $V(\mathbf{x} - \mathbf{y}) = g_0\delta(\mathbf{x} - \mathbf{y})$, where $g_0 < 0$ is the attractive strength of the s-wave coupling.

Path integral description of a Fermi gas (2/4)

How to treat the quartic interaction term $\sim \psi^4$?

- We use a Hubbard-Stratonovich transformation, introducing the auxiliary field $\Delta(x)$ and the shorthand $x = (\mathbf{x}, \tau)$.
- The interaction between fermions is described in terms of an exchange boson.

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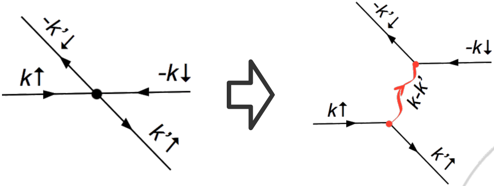

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$$e^{-g_0 \int \mathrm{d}x \; \bar{\psi}_\uparrow(x)\bar{\psi}_\downarrow(x)\psi_\downarrow(x)\psi_\uparrow(x)} \propto \int \mathcal{D}\Delta\mathcal{D}\bar{\Delta} \; e^{\int \mathrm{d}x \; \left(\frac{|\Delta|^2}{g_0} + \bar{\Delta}\psi_\downarrow\psi_\uparrow + \Delta\bar{\psi}_\uparrow\bar{\psi}_\downarrow\right)}$$

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Remarks:

- Essentially a Gaussian integral.
- Physical meaning of the transformation: $\Delta \sim \psi\psi$, as in the BCS theory a finite expectation value signals pairing.
- Result: the quartic interaction is decoupled, but we have introduced a new field, hopefully we can treat in perturbatively.

Path integral description of a Fermi gas (3/4)

After the H/S transformation the partition function can be recast in an elegant way using the Nambu-Gor'kov spinors ($\Psi(x) = \begin{pmatrix} \psi_\uparrow(x) & \bar{\psi}_\downarrow(x) \end{pmatrix}^T$)

$$\mathcal{Z} = \int \mathcal{D}\Delta\mathcal{D}\bar{\Delta}\mathcal{D}\psi_\sigma\mathcal{D}\bar{\psi}_\sigma \exp\left[\int \mathrm{d}x \left(\bar{\Psi}(x)\left[-\mathbb{G}^{-1}\right]_x \Psi(x) - \frac{|\Delta(x)|^2}{g_0}\right)\right]$$

The integration over the fermionic fields ψ_σ, $\bar{\psi}_\sigma$ can now be carried out exactly, being the action quadratic form in the fermionic fields, yielding:

$$\mathcal{Z} = \int \mathcal{D}\Delta\mathcal{D}\bar{\Delta} \exp\left[\mathrm{Tr}\ln\left(-\mathbb{G}^{-1}\right) + \int \mathrm{d}x \frac{|\Delta|^2}{g_0}\right]$$

The complete physics of the system is encoded in the Green's function $\mathbb{G}$.

$$\left[-\mathbb{G}^{-1}\right]_x = \begin{pmatrix} \hbar\partial_\tau + \xi & -\Delta(x) \\ -\bar{\Delta}(x) & \hbar\partial_\tau - \xi \end{pmatrix}$$

with $\xi = -\frac{\hbar^2\nabla^2}{2m} - \mu$.

Path integral description of a Fermi gas (4/4)

How to tackle the problem? Idea: separate a leading (and analytically treatable) contribution from a small contribution (to be treated perturbatively).

We expand the pairing field Δ around a constant and uniform saddle-point (mean-field) configuration Δ_0, as

$$\Delta(x) = \Delta_0 + \eta(x)$$

Mean field

Fluctuations

it follows that

$$\underbrace{\begin{pmatrix} \hbar\partial_\tau + \xi & -\Delta(x) \\ -\bar{\Delta}(x) & \hbar\partial_\tau - \xi \end{pmatrix}}_{[-\mathbb{G}^{-1}]_x} = \underbrace{\begin{pmatrix} \hbar\partial_\tau + \xi & -\Delta_0 \\ -\bar{\Delta}_0 & \hbar\partial_\tau - \xi \end{pmatrix}}_{\left[-\mathbb{G}_{\text{sp}}^{-1}\right]_x} + \underbrace{\begin{pmatrix} 0 & -\eta(x) \\ -\bar{\eta}(x) & 0 \end{pmatrix}}_{[\mathbb{F}]_x}$$

Mean field and fluctuations (1/2)

Expanding the fluctuations up to Gaussian order, similarly we get a mean-field and a Gaussian-level partition function:

$$\mathcal{Z} \approx \int \mathcal{D}\Delta\mathcal{D}\bar{\Delta}\, e^{\mathrm{Tr}\ln\left(-\mathbb{G}^{-1}\right)} = \int \mathcal{D}\Delta\mathcal{D}\bar{\Delta}\, e^{\mathrm{Tr}\ln\left(-\mathbb{G}_{\mathrm{sp}}^{-1}\right)} e^{\mathrm{Tr}\ln\left(\mathbb{1}-\mathbb{G}_{\mathrm{sp}}\mathbb{F}\right)} = \mathcal{Z}_{\mathrm{mf}}\,\mathcal{Z}_{\mathrm{fl}}$$

with:

$$\mathcal{Z}_{\mathrm{mf}} = \det\left(-\mathbb{G}_{\mathrm{sp}}^{-1}\right) \qquad \mathcal{Z}_{\mathrm{fl}} = \int \mathcal{D}\eta\mathcal{D}\bar{\eta}\; e^{-\frac{1}{2}\mathrm{Tr}\left(\mathbb{G}_{\mathrm{sp}}\mathbb{F}\mathbb{G}_{\mathrm{sp}}\mathbb{F}\right)+\int \mathrm{d}x \frac{|\eta|^2}{g_0}}$$

Mean field and fluctuations (2/2)

Using $\mathcal{Z} = e^{-\beta\Omega}$, where Ω is the thermodynamic grand potential, one gets the mean-field equation of state:

$$\Omega_{\mathrm{mf}}(\mu) = \det\left(-\mathbb{G}_{\mathrm{sp}}^{-1}\right)$$

The Gaussian-level contribution to the grand potential ($Q = (\mathbf{q}, \mathrm{i}\Omega_n)$ and Ω_n are Bose Matsubara frequencies.):

$$\Omega_{\mathrm{fl}}(\mu, \Delta_0) = \frac{1}{2\beta} \sum_Q \ln\det(\mathbb{M}(Q))$$

Single particle and collective excitations

One finds that in the gas of paired fermions there are two kinds of elementary excitations: fermionic single-particle excitations with energy

$$E_{\mathrm{sp}}(k) = \sqrt{\left(\frac{\hbar^2 k^2}{2m} - \mu\right)^2 + \Delta_0^2}\ ,$$

where Δ_0 is the pairing gap, and bosonic collective excitations with energy

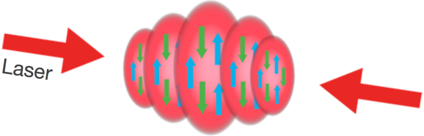
$$E_{\mathrm{col}}(q) = \sqrt{\frac{\hbar^2 q^2}{2m}\left(\lambda\,\frac{\hbar^2 q^2}{2m} + 2mc_s^2\right)}\ ,$$

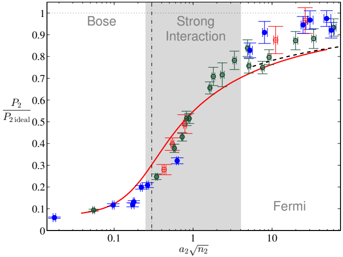
where λ is the first correction to the familiar low-momentum phonon dispersion $E_{\mathrm{col}}(q) \simeq c_s \hbar q$ and c_s is the sound velocity.

The two-dimensional BCS-BEC crossover

The BCS-BEC crossover in 2D (1/2)

An additional laser confinement can be used to create a quasi-2D pancake geometry, trapping the fermions in the antinodes of a standing optical wave.





In 2014 the 2D BCS-BEC crossover has been observed[1] with a **quasi-2D Fermi gas of ^{6}Li atoms** with widely tunable s-wave interaction. The pressure P versus the gas parameter $a_B n^{1/2}$ has been measured.

[1] V. Makhalov, K. Martiyanov, and A. Turlapov, PRL **112**, 045301 (2014).

The BCS-BEC crossover in 2D (2/2)

Why is the 2D case interesting from the theory point of view?

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- The fluctuations are more relevant for lower dimensionalities. The mean field theory can correctly describe (to some extent) the crossover in 3D, we expect it not to work at all in 2D.

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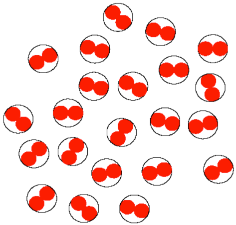
- Qualitatively new physics: a bound state is always present.
- The fluctuations are more relevant for lower dimensionalities. The mean field theory can correctly describe (to some extent) the crossover in 3D, we expect it not to work at all in 2D.
- Berezinskii-Kosterlitz-Thouless mechanism:
    - Mermin-Wagner-Hohenberg theorem: no condensation at finite temperature, no off-diagonal long-range order.
    - Algebraic decay of correlation functions $\langle \exp(i\theta(\mathbf{r})) \exp(i\theta(0)) \rangle \sim |\mathbf{r}|^{-\eta}$
    - Transition to the normal state at a finite temperature T_{BKT}.

The role of Gaussian fluctuations and collective excitations: composite bosons

In the strongly interacting limit an attractive Fermi gas maps to a gas of composite bosons with chemical potential $\mu_B = 2(\mu + \epsilon_B/2)$ and mass $m_B = 2m$. Residual interaction between bosons.

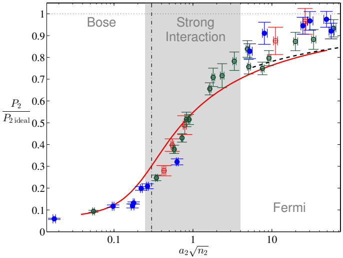


The present theory extends the BCS theory to the strong-coupling regime. One may ask: is the strong coupling limit correctly recovered at mean-field? And at a Gaussian level?

Theory vs. experiments

Equation of state

The pressure measured as a function of the adimensional gas parameter $a_B\sqrt{n_B}$. Experimental data, as shown in the introduction (red curve: smooth approximation of pure 2D Monte Carlo simulation) vs. the present model (gray dashed curve: mean-field, black curve: with fluctuations)



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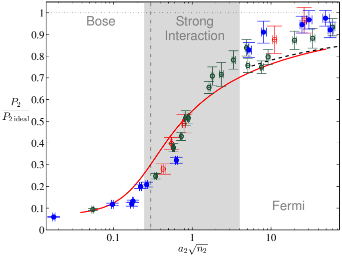

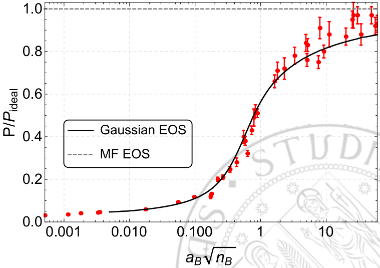

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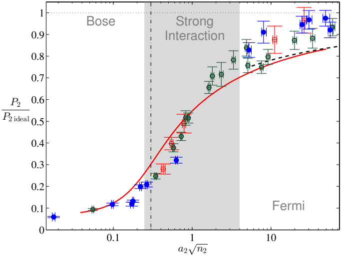
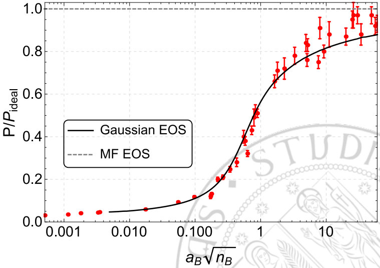




See also: L. He et al., Phys. Rev. A **92**, 023620 (2015).

First sound velocity (1/2)

The first sound velocity c_s can be read from the collective excitations spectrum:

$$E_{col}(q) = \sqrt{\frac{\hbar^2 q^2}{2m}\left(\lambda \frac{\hbar^2 q^2}{2m} + 2m{c_s}^2\right)} \simeq c_s \hbar q$$

The $T = 0$ sound velocity is calculated through the thermodynamics formula:

$$c_s = \sqrt{\frac{n}{m}\frac{\partial \mu}{\partial n}}$$

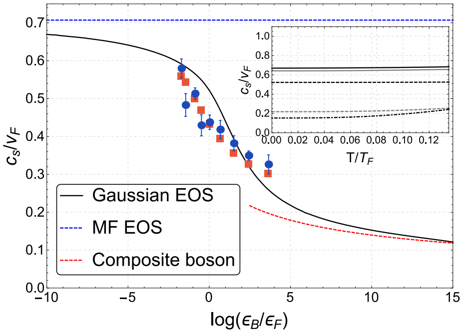
We compare our result with:

- The mean-field result, neglecting Gaussian fluctuations.
- The composite boson limit, obtained through Popov's equation of state for 2D interacting bosons

$$c_s^2 = \frac{4\pi\hbar^2}{m_B^2}\frac{n_B}{\ln\left(\frac{1}{n_B a_B^2}\right)}$$

- Preliminary experimental data (University of Hamburg).

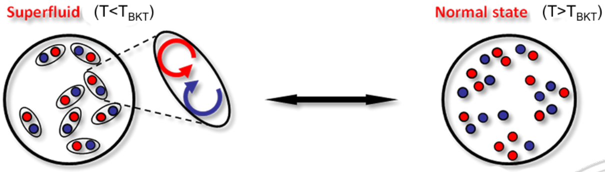
First sound velocity (2/2)



- Away from the weak-coupling limit, in the intermediate region and in the BEC limit the sound velocity c_s is strongly affected by the Gaussian contribution to the equation of state.
- Strong coupling: composite boson limit.
- Quite good agreement with (preliminary) experimental data.
- The temperature dependence (inset) is very weak.

BKT critical temperature (1/4)

The Berezinskii-Kosterlitz-Thouless (BKT) transition separated the low-temperature phase characterized by bound vortex-antivortex pairs from the high-temperature phase characterized by a proliferation of free vortices.



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The BKT critical temperature is found using the Kosterlitz-Nelson (KN) condition:

$$k_B T_{BKT} = \frac{\hbar^2 \pi}{8m} n_s(T_{BKT})$$

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The superfluid density is obtained using Landau's quasiparticle excitations formula for fermionic and bosonic excitations:

$$n_{n,f} = \beta \int \frac{\mathrm{d}^2 k}{(2\pi)^2} k^2 \frac{e^{\beta E_k}}{(e^{\beta E_k} + 1)^2} \qquad \text{and} \qquad n_{n,b} = \frac{\beta}{2} \int \frac{\mathrm{d}^2 q}{(2\pi)^2} q^2 \frac{e^{\beta E_{\mathrm{col}}(q)}}{(e^{\beta E_{\mathrm{col}}(q)} - 1)^2} \,,$$

then $n_s = n - n_{n,f} - n_{n,b}$.

BKT critical temperature (2/4)

Main approximation

The single-particle and collective contributions are not independent, as there is hybridization due to Landau damping at finite temperature.

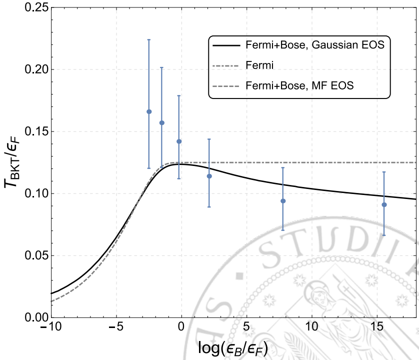
The effect of hybridization is most prominent at $T \sim \epsilon_F$, here in the superfluid phase, below T_{BKT}, one has $k_B T \lesssim 0.125\epsilon_F$ and the hybridization can be safely ignored.

Previous results *a posteriori* confirm that hybridization should be neglectable.

BKT critical temperature (3/4)

We can compare the theory with recently obtained experimental data[1]:

- The agreement with experimental data is very good in the intermediate and strongly coupled regimes.
- The agreement for two points in the weakly-coupled regime is not as good, but still within 1.2σ.
- However, under very general assumptions, $T_{BKT} \lesssim 0.125\epsilon_F$ if the Kosterlitz-Nelson condition holds.

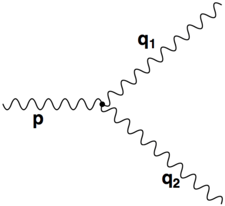


[1] P. A. Murthy et al., Phys. Rev. Lett. **115**, 010401 (2015).

Beliaev damping of collective modes in atomic Fermi superfluids

Beliaev decay: an improved treatment (1/5)

- Beliaev damping in a superfluid is the decay of a collective excitation into two lower-frequency collective excitations.
- The only decay mode for a bosonic collective excitation in the low momentum, $T = 0$ limit.



- **Idea:** develop the theory for a bosonic system, and the apply it to the case of collective excitations in a Fermi superfluid.
- We extend the original, *"linear"* treatment due to Landau allowing for a realistic spectrum for collective excitations.

Main reference: G. Bighin, L. Salasnich, P.A. Marchetti and F. Toigo, Phys. Rev. A **92**, 023638 (2015).

Beliaev decay: an improved treatment (2/5)

The starting point is Landau's hydrodynamic theory of a superfluid. Initially introduced on semi-phenomenological grounds, was then rigorously rederived from the microscopical theory by Popov.

$$\hat{H} = \int \mathrm{d}^3 x \left[\frac{1}{2} \hat{\mathbf{v}} \cdot \hat{\rho} \hat{\mathbf{v}} + \hat{\rho} e(\hat{\rho}) \right]$$

where e is the internal energy per unit mass, $\hat{\mathbf{v}} = \nabla \hat{\phi}$ and $\hat{\rho} = \rho + \hat{\rho}'$ The new operators can be written expanding in plane waves:

$$\hat{\rho}' = \frac{1}{\sqrt{2V}} \sum_{|\mathbf{k}| \neq 0} A_{\mathbf{k}} \left(\hat{b}_{\mathbf{k}} e^{\mathbf{ik}\cdot\mathbf{r}} + \hat{b}^{\dagger}_{\mathbf{k}} e^{-\mathbf{ik}\cdot\mathbf{r}} \right) \qquad \hat{\phi} = \frac{1}{\sqrt{2V}} \sum_{|\mathbf{k}| \neq 0} \mathrm{i}\hbar B_{\mathbf{k}} \left(b_{\mathbf{k}} e^{\mathbf{ik}\cdot\mathbf{r}} - b^{\dagger}_{\mathbf{k}} e^{-\mathbf{ik}\cdot\mathbf{r}} \right)$$

and we impose that $\hat{\rho}'$ and $\hat{\phi}$ should be canonically conjugate variables

$$[\hat{\phi}(\mathbf{r}), \hat{\rho}'(\mathbf{r}')] = -\mathrm{i}\hbar \delta(\mathbf{r} - \mathbf{r}')$$

Beliaev decay: an improved treatment (3/5)

Landau's treatment

Internal energy

$$e(\hat{\rho}) = \frac{u^2 \hat{\rho}^2}{2\rho}$$

By diagonalizing the Hamiltonian one gets the linear spectrum

$$\omega_{\mathbf{k}} = u\hbar k$$

Kinematic constraints imply that the Beliaev decay final states must be collinear to the initial state.

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Present work

Internal energy

$$e(\hat{\rho}, \nabla\hat{\rho}) = e(\hat{\rho}) + \lambda \frac{\hbar^2}{8m^2} \frac{(\nabla\hat{\rho})^2}{\hat{\rho}^2}$$

By diagonalizing the Hamiltonian one gets the Bogoliubov-like spectrum

$$\omega_{\mathbf{k}} = u\hbar k \sqrt{1 + \lambda \frac{\hbar^2}{4m^2} \frac{k^2}{u^2}}$$

The decay angle is determined by λ. For $\lambda = 0$ the original *"linear"* theory is recovered.

Beliaev decay: an improved treatment (4/5)

The Hamiltonian has terms with any number of field operators, the relevant, third-order part is:

$$\hat{H}^{(3)} = \int \mathrm{d}^3 r \Big[(\nabla\hat{\phi}) \frac{\hat{\rho}'}{2} (\nabla\hat{\phi}) + \frac{1}{6} \left(\frac{\mathrm{d}}{\mathrm{d}\rho} \frac{u^2}{\rho} \right) \hat{\rho}'^3 - \lambda \frac{\hbar^2}{8m^2} (\nabla\hat{\rho}')^2 \frac{\hat{\rho}'}{\rho^2} \Big]$$

and the Beliaev decay rate is calculated by taking the matrix element

$$H_{if}^{(3)} = \langle i | H^{(3)} | f \rangle$$

between the following initial and final states

$$|i\rangle = \hat{b}^\dagger_{\mathbf{p}} |\Omega\rangle \qquad |f\rangle = \hat{b}^\dagger_{\mathbf{q}_1} \hat{b}^\dagger_{\mathbf{q}_2} |\Omega\rangle$$

and finally using Fermi's golden rule

$$\mathrm{d}w = \frac{2\pi}{\hbar} |H_{if}|^2 \delta(E_f - E_i) \frac{V^2}{(2\pi\hbar)^6} \mathrm{d}^3 q_1 \mathrm{d}^3 q_2$$

Beliaev decay: an improved treatment (5/5)

Using Landau's linear spectrum one gets the following decay rate:

$$w = p^5 \frac{3}{320\pi\rho\hbar^4}\left(1 + \frac{\rho^2}{3u^2}\frac{\mathrm{d}}{\mathrm{d}\rho}\frac{u^2}{\rho}\right)^2$$

Including the gradient term in the internal energy e and taking into account the full Bogoliubov-like spectrum, one gets

$$w = \frac{9}{32\pi\rho\hbar^4}\int_0^p q^2|p^2+q^2-2pq|^2\cos\theta_0 \frac{\left(1+\chi\frac{\rho}{u^2}\frac{\mathrm{d}}{\mathrm{d}\rho}\frac{u^2}{\rho}\right)^2}{|f'(\cos\theta_0,p,q)|}\mathrm{d}q$$

where $f(\cos\theta, p, q) = \frac{1}{u}\frac{|\mathbf{p}-\mathbf{q}|}{pq}\left(\omega_p - \omega_q - \omega_{|\mathbf{p}-\mathbf{q}|}\right)$, $\theta_0 = \theta_0(p,q)$ is the only zero of f, $\chi^{-1} = (p-q)/\left|\mathbf{p}-\mathbf{q}\right|(1+\cos(\theta_0)) + \cos(\theta_0)$.

Beliaev damping of collective modes in atomic Fermi superfluids (1/4)

We can apply the theory to the collective excitations in a Fermi gas, which are described by the following Bogoliubov-like spectrum

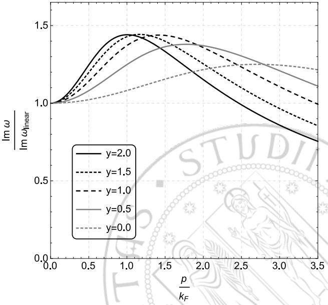
$$E_{col}(q) = \sqrt{\frac{\hbar^2 q^2}{2m}\left(\lambda \frac{\hbar^2 q^2}{2m} + 2m{c_s}^2\right)} \simeq c_s \hbar q$$

exactly mapping on the bosonic spectrum we obtained from the quantum hydrodynamics.

Beliaev damping of collective modes in atomic Fermi superfluids (2/4)

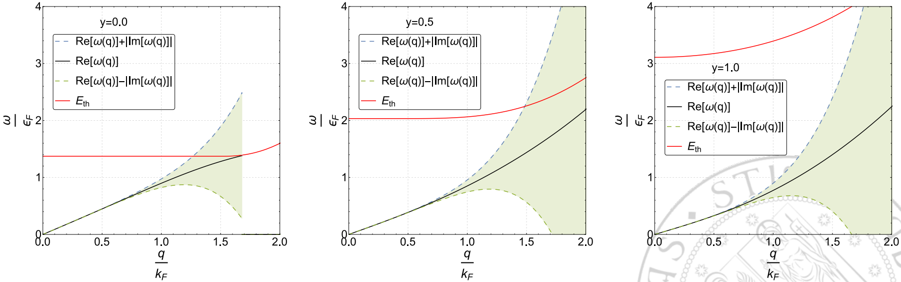
The Beliaev decay is allowed only if the spectrum grows more than linearly, for the collective excitations in the BCS-BEC crossover this means that $y = (k_F a_s)^{-1} \gtrsim -0.14$.

- We calculate the Beliaev decay rate in the BEC side of the BCS-BEC crossover ($y = (k_F a_s)^{-1} \gtrsim 0$) and study the ratio between the present result and Landau's linear theory.
- The corrections to the decay rate when introducing a non-linear, realistic spectrum can be quite relevant, up to 40% w.r.t. Landau's result.



Beliaev damping of collective modes in atomic Fermi superfluids (3/4)

Collective modes spectra for $y = 0.0$, $y = 0.5$, $y = 1.0$. **Black line:** real part. **Green line:** $\pm$ imaginary part. **Red line:** dissociation threshold, the bosonic excitation breaks into two fermionic excitations.



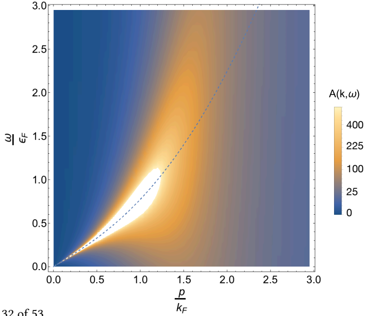
By matching the real and imaginary parts of the spectrum we identify a cutoff scale after which the collective excitation is no longer well defined due to Beliaev decay.

Beliaev damping of collective modes in atomic Fermi superfluids (4/4)

Pair fluctuations spectral function

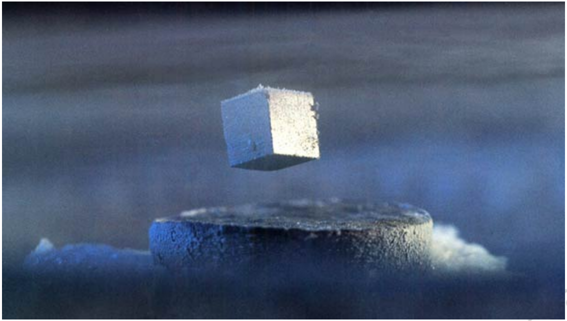
$$A_{\eta\eta}(\mathbf{k},\omega) = -2\,\mathrm{Im}\,G_{\eta\eta}(\mathbf{k},\omega+\gamma_{\mathbf{k}})$$

where G is the Green function, i.e. the inverse of $\mathbb{M}$ seen before, $\gamma_{\mathbf{k}}$ is the imaginary part of the spectrum due to Beliaev decay.



- Most of the spectral weight is concentrated for $p \lesssim k_F$.
- Well before hitting the dissociation threshold the excitation is no longer well defined due to the high decay rate.

High-T_c superconductivity in cuprates



Main references: P. A. Marchetti, F. Ye, Z. B. Su, and L. Yu Phys. Rev. B **84**, 214525 (2011) (for the theoretical framework), P.A. Marchetti and GB, Europhys. Lett. **110**, 37001 (2015) (for the superfluid density).

Cuprates: an overview (1/2)

- **Superconducting cuprates**: a class of superconducting materials with very high critical temperatures (up to 135 °K), characterized by CuO_2 planes.
- Discovered in 1986 by J. G. Bednorz e K. A. Müller; Nobel prize awarded in 1987, the fastest in history.
- Very active research field: more than 100,000 research articles in $\sim$ 25 years.
- To date the microscopical mechanism behind SC in cuprates has not yet been completely understood.

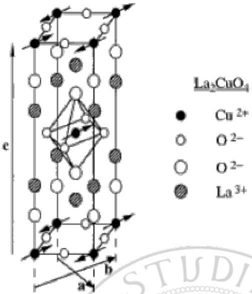
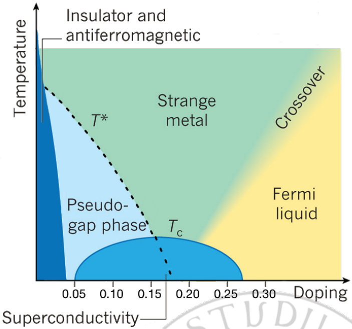


Figure: Unitary cell for La_2CuO_4.

Cuprates: an overview (2/2)

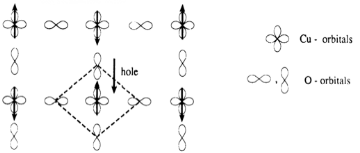
- Different chemical compositions (YBCO, LSCO, BSSCO) the only common chemical features being the CuO_2 planes.
- As a consequence the CuO_2 planes are believed to be the main seat of superconductivity.



- The onset of superconductivity is controlled by the doping (additional holes injected into the CuO_2 planes) and by temperature. Universal **phase diagram**.
- BCS theory can not account for SC in cuprates.

From the CuO_2 planes to the $t - J$ model

CuO_2 planes in terms of Zhang-Rice singlets:



ZR: Doping-induced hole reside (primarily) on combinations of four oxygen p orbitals centered around a copper site. Antiferromagnetic background.

From ZR singlets to the $t - J$ model:

- Strong on-site repulsion (Gutzwiller projector P_G)
- Nearest neighbour hopping ($t \approx 0.3$ eV)
- Antiferromagnetic Heisenberg term ($J \approx 0.1$ eV)

$$H_{t-J} = \sum_{\langle i,j \rangle} P_G \left[-t \sum_{\alpha} c^\dagger_{i\alpha} c_{j\alpha} + h.c. + J \mathbf{S}_i \cdot \mathbf{S}_j \right] P_G$$

"Doping a Mott insulator", P.A. Lee, N. Nagaosa, X.-G. Wen, Rev. Mod. Phys. **78**, 17 (2006)

Spin-charge separation

The electron creation/annihilation operators are decomposed as follows:

$$\hat{c}_{i,\alpha} = \hat{s}_{i,\alpha} \hat{h}_i^\dagger$$

where:

- $\hat{h}_i$ is a spinless fermion (holon): the P_G constraint is automatically satisfied due to Pauli exclusion principle.
- $\hat{s}_{i,\alpha}$ is a spin $1/2$ boson (spinon).

A local invariance introduced:

$$U(1)_{h/s} \qquad \begin{cases} \hat{s}_{i,\alpha} \longrightarrow \hat{s}_{i,\alpha} e^{\mathrm{i}\phi(x)} \\ \hat{h}_i \longrightarrow \hat{h}_i e^{\mathrm{i}\phi(x)} \end{cases}$$

Emergent $U(1)$ gauge field: $A_\mu \approx s^*_\alpha \partial_\mu s_\alpha + \cdots$

Holons, spinons and statistical fluxes (1/2)

In two dimensions one can bind a statistical flux Φ to holons and spinons, modifying the statistics:

$$c_{j\alpha} = e^{-\mathrm{i}\Phi_h(j)}\, h_j^* \left(e^{\mathrm{i}\Phi_s(j)}\, s_j\right)_\alpha$$

Example: Jordan-Wigner transformation 1D

Analogously in 1D

$$c_j^\dagger \longrightarrow a_j^\dagger\, e^{-\mathrm{i}\pi \sum_{l<j} a_l^\dagger a_l}$$

where $c^\dagger$ is a fermionic operator, $a^\dagger$ is a bosonic one and the additional phase restores the correct statistics.

Holons, spinons and statistical fluxes (1/2)

In two dimensions one can bind a statistical flux Φ to holons and spinons, modifying the statistics:

$$c_{j\alpha} = e^{-\mathrm{i}\Phi_h(j)} h_j^* \left(e^{\mathrm{i}\Phi_s(j)} s_j\right)_\alpha$$

In 2D we extend the idea behind Chern-Simons bosonization: just like in the 1D case one can bind a "string" to an excitation modifying the statistics.
However in 2D an additional gauge field needs to be minimally coupled and an additional "kinetic" term for the gauge field is also needed. The coefficient of this kinetic term regulates the statistics of the excitation+flux combination.

Holons, spinons and statistical fluxes (2/2)

This new representation of the $t - J$ model in terms of holons, spinons and statistical fluxes is completely equivalent to our starting point. One can tune the statistics of holons and spinons as long as the recomposed hole is still a fermion. We choose the operator+flux combination to be a **semion**, i.e. acquiring a $\pm$i factor upon exchange: why do we divide the electron degrees of freedom in this way?

- The charge and spin degrees of freedom can be decomposed in many different ways (e.g. slave boson/slave fermion as particular cases), however the MF results are very different.
- In 1D case a semionic theory reproduces the correct critical exponents.

Anomalous (semionic) statistics possible because the system is 2D (like the FQHE).

Effective action

Mean-field approximation: neglect the holon fluctuations in Φ_h and the spinon fluctuations in Φ_s.

Effective action for the model:

- Holons are (formally) relativistic Dirac fermions.
- Spinons are described by a non-linear massive σ model, with $m \sim |\delta \ln \delta|$.
- The gauge field A_μ is minimally coupled to holons and spinons: it corresponds to the h/s symmetry: it provides a *"gauge glue"* between the two components of the electron.

Final result: we have an effective description of the $t - J$ model in terms of holons and spinons.

Towards superconductivity

The electron has a composite structure:
spinon + holon

⇨

Superconductivity is achieved in three stages:

- Holon pairing
- Spinon pairing
- Phase coherence

The pairing process

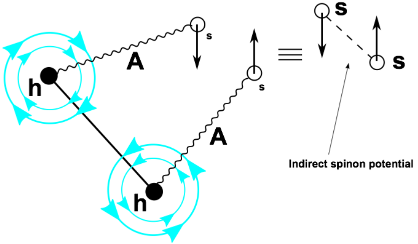


Figure: The attractive potential between the spinons, essential for the SC, is mediated by a gauge field "binding" holon and spinons, and by the holon attraction.

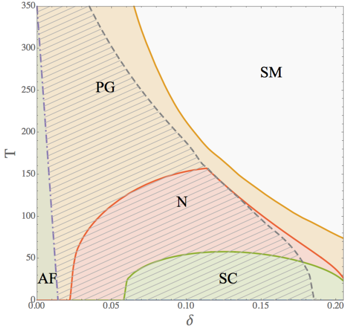
The superconductivity is achieved in **three steps**: holon pairing (T_{ph}), spinon pairing (T_{ps}), phase coherence (T_c):

$$\Delta_c \sim \frac{|\Delta^s|}{|\Delta^h|} e^{i\overbrace{(\phi_s - \phi_h)}^{\equiv \phi}}$$

$$SC \Longleftrightarrow \langle \Delta_c \rangle \neq 0$$

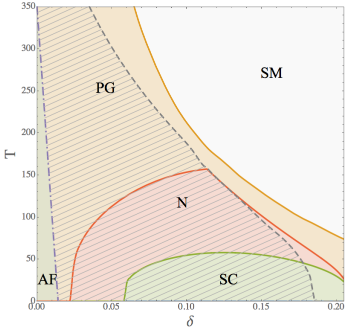
The direct spinon-spinon interaction is repulsive, the gauge fluctuations play a key role.
The phase dynamics are described by a gauged XY model, so that the SC transition is essentially XY.

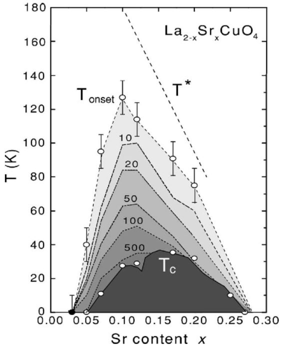
Phase diagram



T_{ph} (yellow line) is the holon pairing temperature. T_{ps} (red line) is is the spinon pairing temperature and encloses the N region in which the system supports a Nernst signal. The crossover PG-SM is denoted by the dashed line. T_c (green line) is determined from the transition temperature of the XY model of spinons. The Nèel temperature (dot-dashed line), delimiting the region characterized by anti-ferromagnetic (AF) order, is qualitative from experiments.

Phase diagram





From Y. Wang, L. Li, N.P. Ong, Phys. Rev. B **73**, 024510 (2006).

Superfluid density

The superfluid density ρ_s is one of the main signatures of the superconducting transition.

- It can be defined as the coefficient governing phase fluctuations in an effective action for superconductivity:

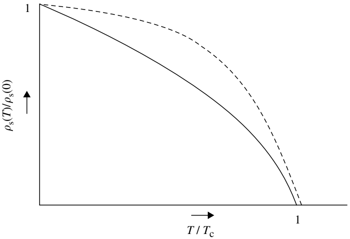
$$S_{\text{eff}} = \frac{\rho_s}{2} \int \mathrm{d}\tau \mathrm{d}^d r \, (\nabla\theta)^2 + \cdots$$

- It is simply related to the Meissner effect and to the London penetration depth:

$$\lambda = \sqrt{\frac{m}{\mu_0 \rho_s e^2}}$$

Superfluid density in cuprates

A great deal of data is available for superfluid density in cuprates. The peculiar behaviour of ρ_s is very different from that of conventional materials, and it represents **a longstanding puzzle**.

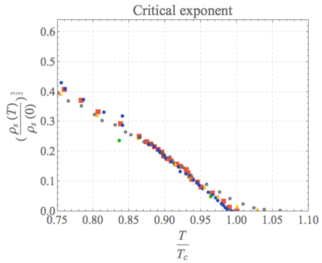


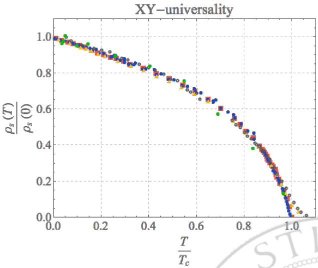
It exhibits a combination of features that do not allow for a simple explanation (within BCS-like or XY-like models). In the figure: cuprates (solid) vs. s-wave BCS (dashed line).

	Well-defined gapped Fermi arcs (ARPES)	ρ_s (near $T = 0$) is linear in T	Critical exponent $\frac{2}{3}$	Uemura relation $\rho_s(T = 0) \propto T_c$
BCS	✓	✓	✗ (would be 1)	✗
3DXY	✗	✓ (but slope?)	✓	? (it depends...)

Experimental review

A review of experimental data in moderately underdoped and optimally doped cuprates: critical exponent 2/3 and universality in renormalized superfluid density.





- YBCO, x=6.95, a axis, from Kuan Zhang et al., Phys. Rev. Lett. 74, 1008 (1995)
- Universality in YBCO, x=6.6, x=6.95, x=6.99, a axis, from D.A. Bonn et al., Czechoslovak J. of Phys. 46, 3195 (1996)
- Bi−2212, λ_{ab}=3000 Å, from Jacobs et al., Phys. Rev. Lett. 75 4516 (1995)
- LSCO, x=0.10, from Panagopoulos et al., Phys. Rev. B. 69, 14617
- Hg−1201, x=0.10, from Panagopoulos et al., Phys. Rev. B. 69, 14617

Experimental review

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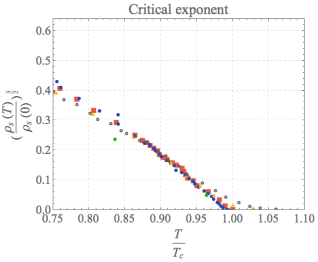
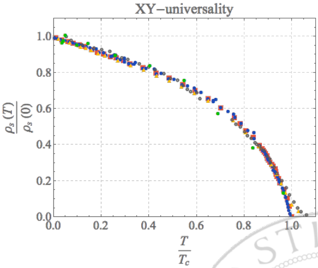




Can we interpret these experimental data within the present formalism?

A gauge approach to superfluid density (1/2)

In the pseudogap (PG) regime, according to the present model:

- The holon contribution is a "standard" BCS-like d-wave superfluid density.

$$\rho_{s,h}(T) = \frac{2\epsilon_F}{\pi}\left(1 - \frac{\log(2)}{2\Delta_h}T\right)$$

- The contribution from spinons is that of a 3DXY model (ρ_{XY}), with effective temperature $\Theta = 3\pi(m_s - \Delta_s^2/m_s)/\Delta_s^2$.

$$\rho_{s,s}(T) = \xi\left[\frac{\mathrm{d}\Theta}{\mathrm{d}T}(0)\right]^{-1}\rho_{XY}(\Theta(T)/\Theta(T_c))$$

- The superfluid densities sum according the Ioffe-Larkin rule, due to the gauge string between holons and spinons. (in PG $\rho_{s,s} \ll \rho_{s,h} \implies \rho_s \approx \rho_{s,s}$)

$$\rho_s = \frac{\rho_{s,s}\rho_{s,h}}{\rho_{s,s} + \rho_{s,h}}$$

A gauge approach to superfluid density (1/2)

Our solution to the puzzle...

- The critical properties near T_c are determined by the spinons, whose dynamics is 3DXY-like ($\rightarrow 2/3$ critical exponent)
- The Fermi surface, on the other hand, is determined by the holon BCS-like part of the theory ($\rightarrow$ possibility of Fermi arcs).
- Finally the superfluid density in the moderate underdoping regime is dominated by the spinon contribution ($\rightarrow$ 3DXY behaviour of normalized $\rho_s(T/T_c)$ and Uemura relation).

Comparison with experimental data (1/2)

The model can accurately fit normalized superfluid density data from moderate underdopings to optimal doping with only one $\mathcal{O}(1)$ free parameter, regulating the relative weight of holons and spinons.

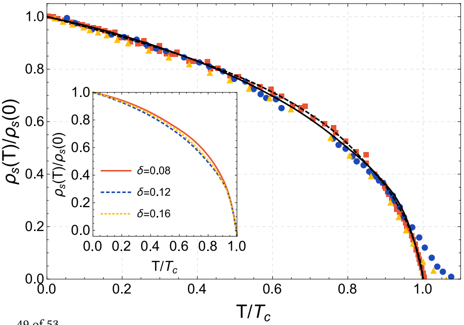


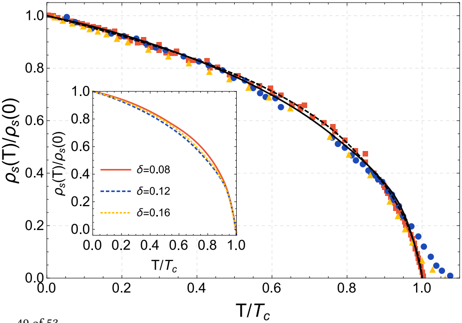
Figure: The normalized superfluid density vs. $\frac{T}{T_c}$. Our theoretical calculation (solid line, $\delta = 0.12$) is compared with a pure 3D XY model (dashed line), squares corresponding to the near-universal YBCO behavior for superfluid density, circles for $\delta = 0.075$ LSCO, triangles for near-optimal-doping BSCCO. The near-universal behavior of our ρ_s is shown in the inset.

Comparison with experimental data (1/2)

- The experimentally observed critical behavior is exactly reproduced:

$$\rho_s \sim \left| \frac{T - T_c}{T_c} \right|^{\frac{2}{3}} \quad \text{for } T \longrightarrow T_c$$

- The doping-quasi-universality is also a feature of the model.



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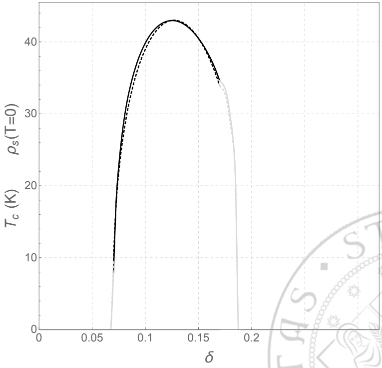
Comparison with experimental data (2/3)

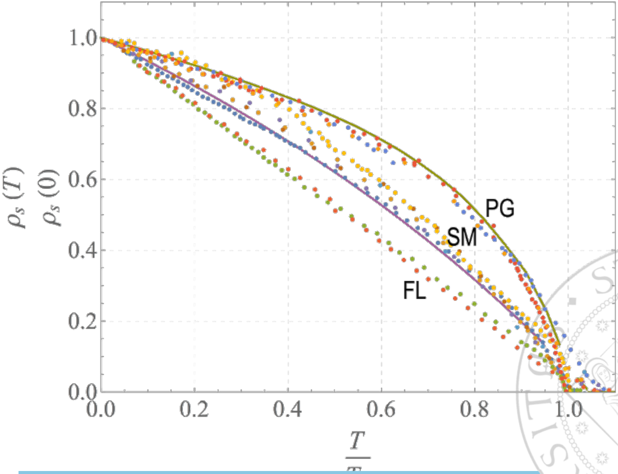
Uemura relation in underdoped cuprates $\rho_s(T=0) \propto T_c$, derived in an (approximate) analytical form.

Figure: Uemura et al. (Phys. Rev. Lett. **62**, 2317 (1989)) observed strong linearity between $\rho_s(T=0)$ and T_c in underdoped cuprates ($0 \leq \delta \leq 0.15$). Theoretically calculated $T=0$ superfluid density (solid line, arbitrary units) and critical temperature (dashed line) vs. δ exhibits an approximate Uemura.



Three universality classes

Three "universality classes" in (normalized) superfluid density in cuprates: PG, SM, FL?



Conclusions

- The theoretical treatment of a 2D Fermi gas needs the inclusion of Gaussian fluctuations.
- The equation of state, the first sound velocity, the BKT critical temperature calculated within this formalism show good agreement with experimental data.
- The Beliaev decay is the only decay mode at $T = 0$ for collective excitation in an ultracold Fermi gas; it has been studied for the collective modes of the BCS-BEC crossover.
- A gauge approach to cuprates correctly fits experimental data from moderately underdoped to optimally doped samples, reproducing the correct critical exponent and the Uemura relation.

Thanks for your attention.